

Modeling and Measuring Systemic Risk

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Motivation and Outline

- Various **financial crises** have highlighted the paramount importance of **systemic risk** in the financial sector.
- The **tremendous cost** of systemic risk requires instruments for an efficient macroprudential regulation of financial institutions.

Topics of the talk

- (i) **Models of systemic risk: a class of network models**
- (ii) **Measures of systemic risk: a multivariate approach**

Part I: Models of Systemic Risk

Motivation

- **Systemic risk**: financial system as a whole is susceptible to failures initiated by the characteristics of the system itself
- **Local** and **global** interaction channels:
direct liabilities, bankruptcy costs, cross-holdings and fire sales
- Provide **fully integrated model**; this is missing in the literature so far
 - Bankruptcy costs are, for example, considered by Rogers & Veraart (2013), Elliott, Golub & Jackson (2014), Elsinger (2009) and Glasserman & Young (2014), cross-holdings e.g. by Elsinger (2009) and Elliott et al. (2014). Cifuentes et al. (2005) incorporate fire sales into the framework of **Eisenberg & Noe (2001)**; their approach is further extended by Gai & Kapadia (2010), Nier, Yang, Yorulmazer & Alentorn (2007), Amini, Filipović & Minca (2013), and Chen, Liu & Yao (2014).
- In **numerical case studies** we will analyze the number of contagious defaults

Outline – Part I: Models of Systemic Risk

- (i) **Comprehensive model of financial network**
 - direct liabilities, bankruptcy costs, cross-holdings, fire sales
- (ii) **Existence of equilibrium and algorithm**
- (iii) **Numerical case studies**

A Financial Network Model

Main Interaction Channels

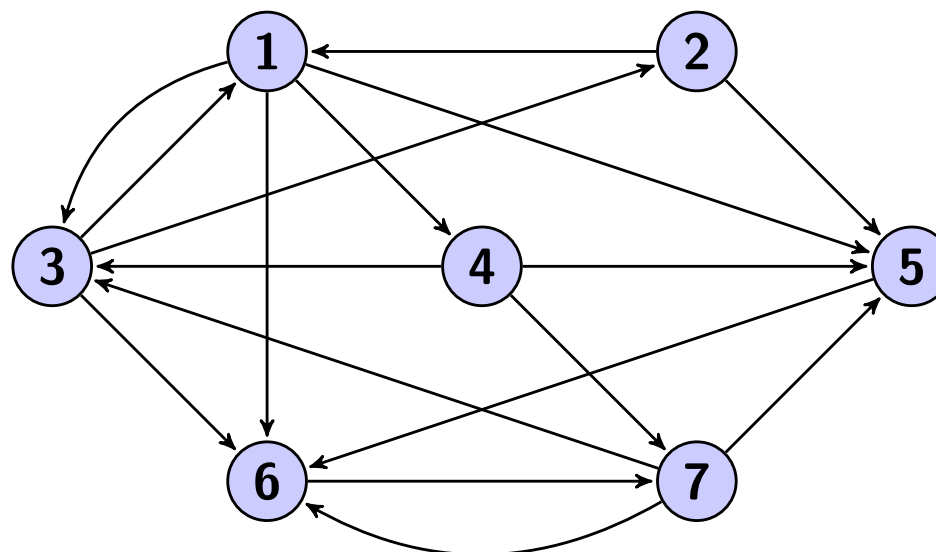
Single period model: **Snapshot** of a banking system that continues to exist afterwards.

Banks are connected to each other via three different channels:

- **Direct liabilities:** Banks have **nominal liabilities against each other**.
- **Fire sales:** If the portfolios of the banks contain the same assets, **changes in asset prices simultaneously influence the net worths** of these banks.
- **Cross-holdings:** Banks may **hold shares of each other**.

In addition, we include **bankruptcy costs**.

Network of Liabilities



Fixed point / equilibrium:

Liability payments

=

minimum of promised payments and available resources

Model Setup

Financial system:

- Banks: $\mathcal{N} = \{1, \dots, n\}$
- Vector of financial net worths of banks: $\mathbf{w} \in \mathbb{R}_+^n$

External assets:

- Cash asset: $r \in \mathbb{R}_+^n$
- Illiquid asset: $s \in \mathbb{R}_+^n$ with price $\mathbf{q} = f(\theta)$
 - θ is the sold quantity of the illiquid asset in the market
- Corresponding total value: $r_i + s_i q, i \in \mathcal{N}$

Liabilities

- **Nominal** liabilities matrix: $L \in \mathbb{R}^{n \times n}$
 - $L_{ij} \geq 0$ interbank obligation of bank i to bank j
- **External** liabilities: $l \in \mathbb{R}_+^n$
- Vector of **total liabilities**: $\bar{p} \in \mathbb{R}_+^n$: $\bar{p}_i = \sum_{j \in \mathcal{N}} L_{ij} + l_i, \quad i \in \mathcal{N}.$
- **Relative liabilities** matrix: $\Pi \in \mathbb{R}^{n \times n}$: $\Pi_{ij} = \begin{cases} L_{ij}/\bar{p}_i, & \text{if } \bar{p}_i > 0 \\ 0, & \text{otherwise.} \end{cases}$
- **Realized payments**: $\mathbf{p} \in \mathbb{R}_+^n$ such that $p_i \leq \bar{p}_i$ for $i \in \mathcal{N}$
- Value of payments:

$$\sum_{j \in \mathcal{N}} \Pi_{ji} p_j.$$

Cross-Holdings

- Cross-holdings matrix: $C \in \mathbb{R}^{n \times n}$
 - $C_{ij} \hat{=}$ fraction of bank i that is held by bank j
 - $C_{ij} \geq 0$, $C_{ii} = 0$, $\sum_{j \in \mathcal{N}} C_{ij} < 1$

- Value of cross-holdings:

$$\sum_{j \in \mathcal{N}} C_{ji} \cdot \max(w_j, 0)$$

- Price impact:
 - A share $\nu_i(\mathbf{p}, \mathbf{w})$ of cross-holdings can be exchanged against the liquid asset to settle liabilities
 - Banks can only realize a fraction of $\lambda_i \in [0, 1]$

Bankruptcy Costs

- Banks that cannot fulfill their total obligations (i.e. $p_i < \bar{p}_i$) are **bankrupt**
- **Legal or administrative expenses** may be incurred
- Two parameters describe bankruptcy costs:
 - Realized fraction of **external asset value**: $0 \leq \alpha \leq 1$
 - Realized fraction of **interbank asset value**: $0 \leq \beta \leq 1$

Price-Payment Equilibrium

A **price-payment equilibrium** is a pair $(p^*, q^*) \in [0, \bar{p}] \times [q_{\min}, q_0] \subseteq \mathbb{R}^{n+1}$, consisting of a *clearing payment vector* p^* and a *clearing price* q^* , such that

$$(\mathbf{p}^*, \mathbf{q}^*) = \Phi(\mathbf{p}^*, \mathbf{q}^*),$$

$$\Phi_i(p, q) := \begin{cases} \chi_i(p, q), & \text{for } i = 1, \dots, n, \\ f(\theta(p, q)), & \text{for } i = n + 1, \end{cases}$$

$$\chi_i(p, q) := \begin{cases} \bar{p}_i, & \text{if } r_i + s_i q + \eta_i(p, q) \geq \bar{p}_i, \\ \alpha[r_i + s_i q] + \beta[\eta_i(p, q)], & \text{otherwise,} \end{cases}$$

$$\eta_i(p, q) := \sum_{j \in \mathcal{N}} \Pi_{ji} p_j + \mu_i(p, q) \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0),$$

$$\mu_i(p, q) = \nu_i(p, q) \lambda_i + 1 - \nu_i(p, q),$$

$$\nu_i(p, q) = \min \left(\frac{\max(\bar{p}_i - r_i - \sum_{j \in \mathcal{N}} \Pi_{ji} p_j - (1 - \mathbb{I}_i) s_i q, 0)}{\lambda_i \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0)}, 1 \right),$$

$$\theta(p, q) := \sum_{i \in \mathcal{N}} \min \left(\frac{\max(\bar{p}_i - r_i - \sum_{j \in \mathcal{N}} \Pi_{ji} p_j - \mathbb{I}_i \lambda_i \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^*(p, q), 0), 0)}{q}, s_i \right).$$

Existence and Computation of Equilibria

- (i) There exist a **unique greatest** and a **unique smallest price-payment equilibrium**, (p^+, q^+) and (p^-, q^-) .
- (ii) The equilibria can be computed in at most $n + 1$ iterations of our **greatest/smallest-clearing-algorithm**.

Algorithm

Set $k = 0$, $(p^{(0)}, q^{(0)}) := (\bar{p}, q_0)$, $\mathcal{D}_{-1} := \emptyset$.

Determine $w^{(0)} := w^*(p^{(0)}, q^{(0)})$ and

$$\mathcal{D}_0 = \{i \in \mathcal{N} \mid w_i^{(0)} < 0\} \quad \text{and} \quad \mathcal{S}_0 = \{i \in \mathcal{N} \mid w_i^{(0)} \geq 0\}.$$

If $\mathcal{D}_0 = \mathcal{D}_{-1}$ and no bank has to liquidate its illiquid asset holdings, i.e. for all $i \in \mathcal{N}$:

$$r_i + \sum_{j \in \mathcal{N}} \Pi_{ji} p_j^{(0)} + \mathbb{I}_i \lambda_i \sum_{j \in \mathcal{N}} C_{ji} \max(w_j^{(0)}, 0) \geq \bar{p}_i,$$

terminate. Otherwise go to Step 2.

Algorithm (2)

Step 1: Determine the sets of defaulting and surviving banks

$$\mathcal{D}_k = \{i \in \mathcal{N} \mid w_i^{(k)} < 0\}, \quad \mathcal{S}_k = \{i \in \mathcal{N} \mid w_i^{(k)} \geq 0\}.$$

If $\mathcal{D}_k = \mathcal{D}_{k-1}$, terminate. Otherwise, go to Step 2.

Step 2: Set $p_i^{(k+1)} = \bar{p}_i$ for all $i \in \mathcal{S}_k$, $p_i^{(k+1)} = x_i$ for all $i \in \mathcal{D}_k$, $q^{(k+1)} = y$ and $w^{(k+1)} = w^*(x, y)$, where (x, y) is determined as the maximal solution to the following system of equations:

$$\begin{aligned} x_i &= \alpha [r_i + s_i y] + \beta \left[\sum_{j \in \mathcal{D}_k} \Pi_{ji} x_j + \sum_{j \in \mathcal{S}_k} [\Pi_{ji} \bar{p}_j + \lambda_i C_{ji} \max(w_j^*(x, y), 0)] \right], \\ y &= f \left(\sum_{i \in \mathcal{D}_k} s_i + \sum_{i \in \mathcal{S}_k} \min \left(\frac{\zeta_i^{(k)}(x, y)}{y}, s_i \right) \right), \\ \zeta_i^{(k)}(x, y) &= \max \left(\bar{p}_i - r_i - \sum_{j \in \mathcal{D}_k} \Pi_{ji} x_j - \sum_{j \in \mathcal{S}_k} [\Pi_{ji} \bar{p}_j + \mathbb{I}_i \lambda_i C_{ji} \max(w_j^*(x, y), 0)], 0 \right), \forall i \in \mathcal{S}_k. \end{aligned}$$

Set $k \rightarrow k + 1$ and go to Step 1.

Numerical Examples

Simulation Methodology (1)

The integrated financial system is characterized by:

$$(\Pi, \bar{p}, r, s, f, \alpha, \beta, \lambda, C)$$

- **Direct liabilities**

- Simulate Π as a random network with fixed characteristics.
- Choose a fixed value for \bar{p} .

- **Fire sales**

- **Bankruptcy costs**

- **Cross-holdings**

Simulation Methodology (2)

The integrated financial system is characterized by:

$$(\Pi, \bar{p}, r, s, f, \alpha, \beta, c, d)$$

- **Fire sales**

- Total external assets e are allocated to liquid and illiquid assets:

$$r = (1 - \rho) \cdot e, \quad s = \rho \cdot e$$

- Inverse demand function: $f(x) = \exp(-\gamma x)$, $\gamma > 0$

→ Fire sales parameters: γ, ρ

- **Bankruptcy costs**

- Realized fraction of external asset value: α

- Realized fraction of interbank asset value: β

→ Bankruptcy costs parameters: α, β

Simulation Methodology (3)

The integrated financial system is characterized by:

$$(\Pi, \bar{p}, r, s, f, \alpha, \beta, \lambda, C)$$

- **Cross-holdings**

- C is generated as an Erdős-Rényi random network:

- * **Diversification** d

- = Expected number of interbank shareholders

- * **Integration** c

- = Fraction of net worth sold as cross-holding shares

- For all $i \in \mathcal{N}$: $\lambda_i = (1 - c)\kappa$, for fixed $\kappa \in [0, 1]$

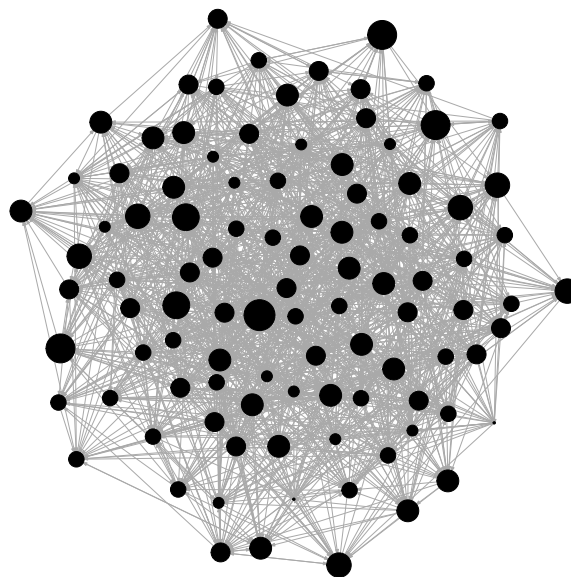
→ Cross-holdings parameters: c, d

Simulation Methodology (4)

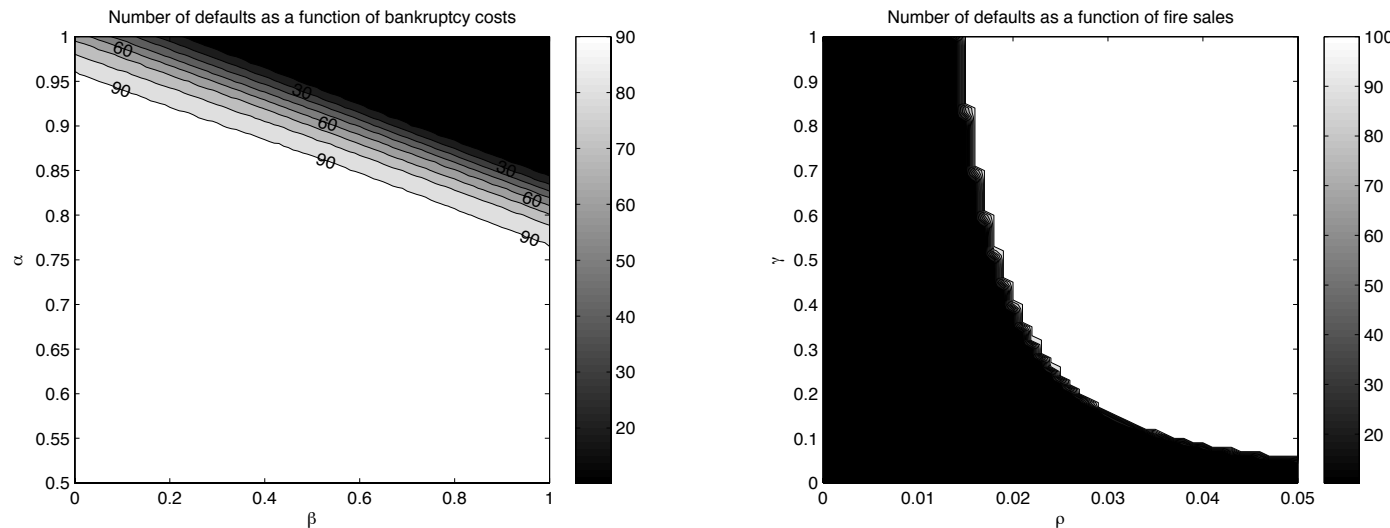
- (i) Choose parameter values $(\alpha, \beta, \gamma, \rho, c, d)$.
- (ii) Simulate direct liabilities Π , cross-holdings C and corresponding external assets.
- (iii) Choose one bank i uniformly at random and set its external assets to zero, i.e. $r_i = s_i = 0$.
- (iv) Calculate the greatest clearing equilibrium and the induced number of defaults.
- (v) Repeat simulations to compute the empirical distribution of the defaults, their average number and their variance.

Erdős-Rényi Networks

$n = 100$, average out-degree: 15



Bankruptcy Costs and Fire Sales

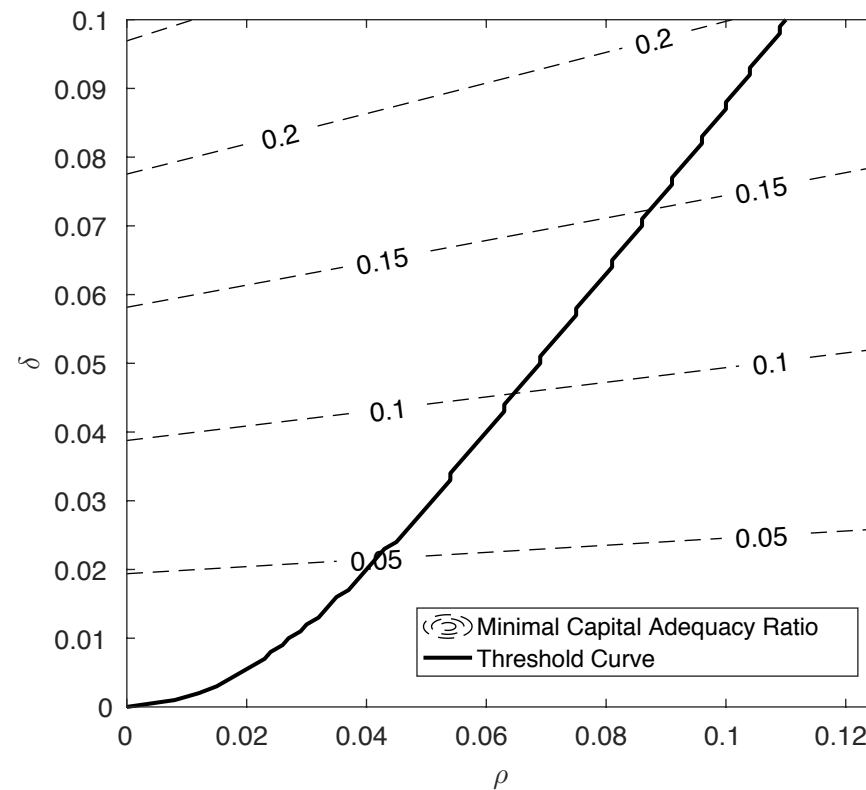


Contour plots of the number of defaults for $n = 100$ banks as a function of (a) bankruptcy costs and (b) fire sales, averaged over 1000 simulations of Π .

Observations and Implications

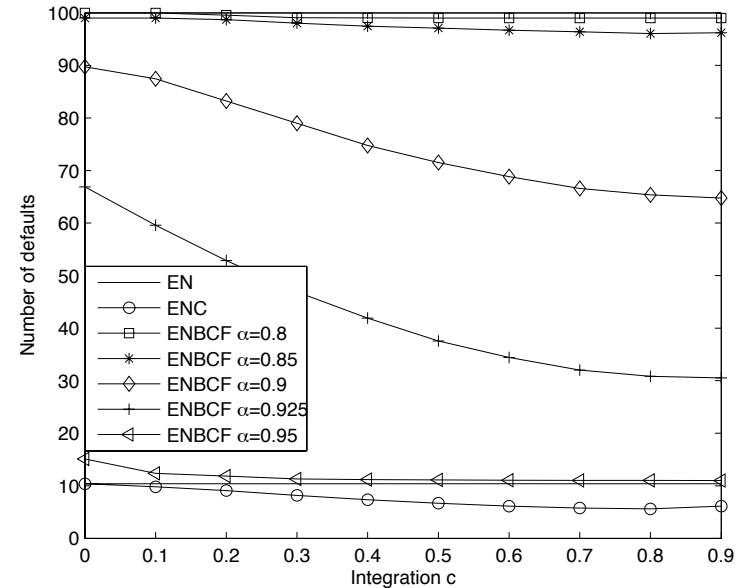
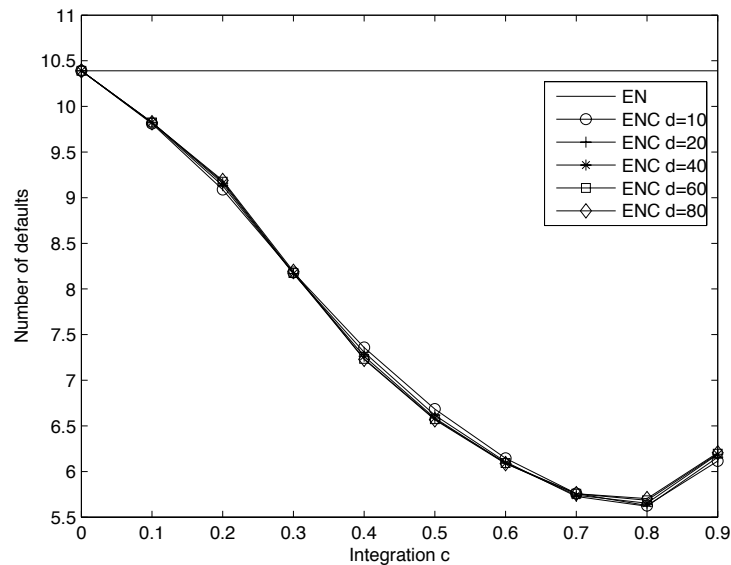
- Sharp boundaries between regimes, concentration on extreme scenarios
- Bankruptcy costs are an important factor. Policies should thus improve the efficiency of managing defaults and restructuring institutions as well as limit the complexity of financial products and the operations of financial institutions. Last wills of institutions could also be a promising instrument.
- The significance of illiquidity emphasizes that providing short-term funding is important. Quantitative easing might be an appropriate instrument for stabilizing the banking sector.

Capital Adequacy Ratios



Level sets of the lowest capital adequacy ratio $CAR := \min_{i \in \mathcal{N}} CAR_i$ as a function of the buffer δ and the proportion ρ of the illiquid asset; the solid line is the boundary between many (lower right) and few (upper left) defaults; averaged over 100 simulations of Π .

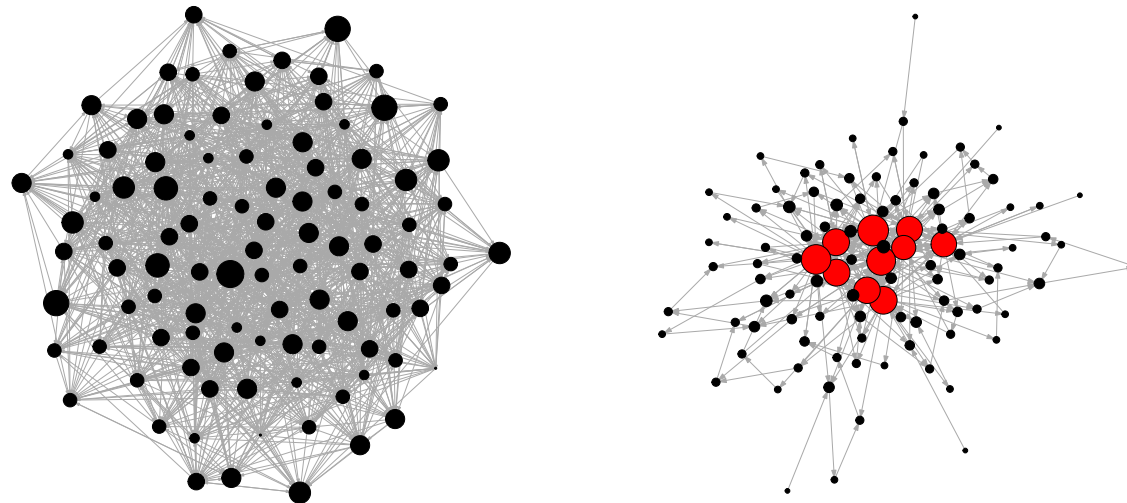
Cross-Holdings: Separate and Joint Effects



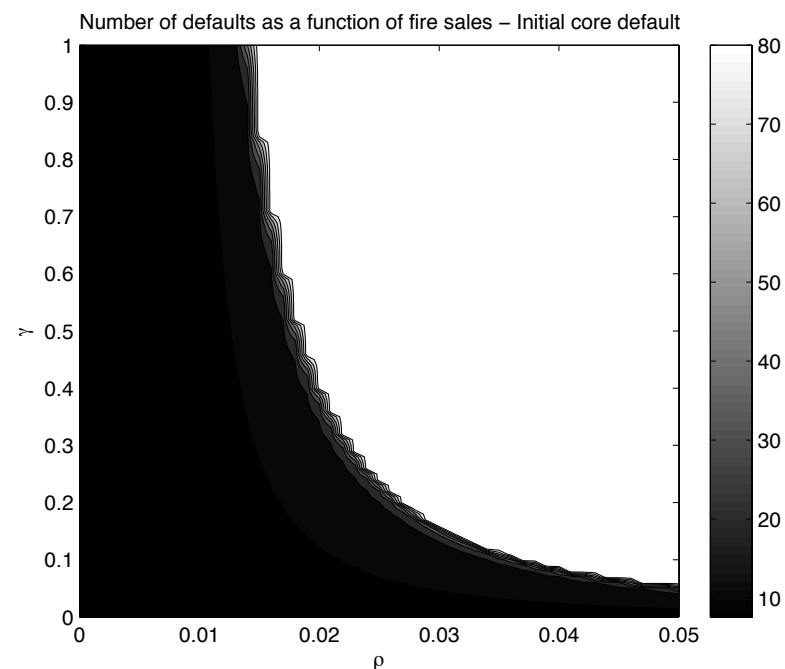
Average number of defaults for $n = 100$ banks as a function of integration of the cross-holdings matrix C .

Erdős-Rényi vs. Core-Periphery Networks

$$n = 100, n^C = 10, n^P = 90$$

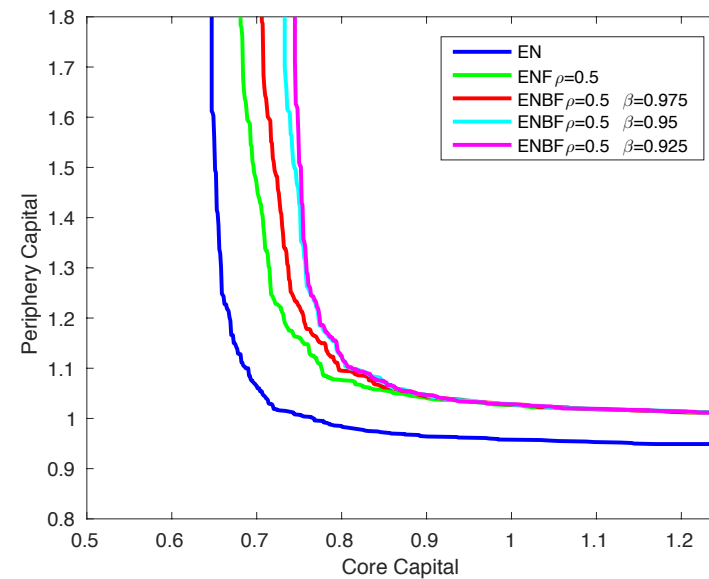
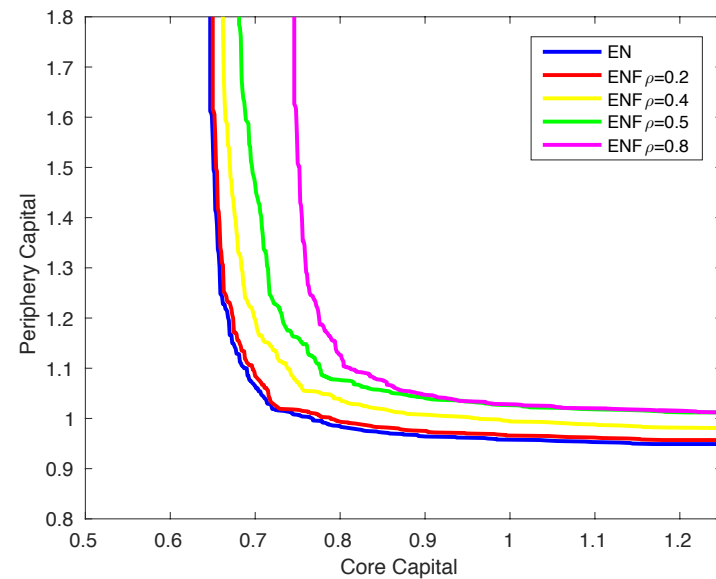


Core-Periphery: Fire Sales



Contour plots of the number of defaults for $n = 100$ banks as a function of fire sales, conditional on an initial core shock.

Measures of Systemic Risk



Capital requirements for core and periphery banks as a function of fire sales.

Part II: Measures of Systemic Risk

Outline – Part II: Measures of Systemic Risk

(i) Measures of systemic risk

- General definition on the basis of acceptance sets and system models

(ii) Cash-flow and value models

- Aggregation functions
- Connection to network models

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Before we focus on these issues, we start with a short

Review of the theory of monetary risk measures

Regulatory and Solvency Capital

- **The role of capital**
 - Buffer for potential losses
 - that protects customers, policy holders and other counterparties
- **The role of the balance sheet**
 - Market-consistent valuation of all assets and liabilities
- **Simple example: Solvency II**
 - $SCR = \text{Solvency Capital Requirement}$
 - Key goal: Limit one-year probability of ruin to at most 0.5%.

SCR

The **SCR** corresponds to the economic capital a (re)insurance undertaking needs to hold in order to limit the probability of ruin to 0.5%, i.e. ruin would occur once every 200 years. . .

The **SCR** is calculated using Value-at-Risk techniques, either in accordance with the standard formula, or using an internal model: all potential losses, including adverse revaluation of assets and liabilities, over the next 12 months are to be assessed. The **SCR** reflects the true risk profile of the undertaking, taking account of all quantifiable risks, as well as the net impact of risk mitigation techniques.

(Proposal for a Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance - Solvency II, COMMISSION OF THE EUROPEAN COMMUNITIES, Brussels, 10.7.2007)

The Balance Sheet

Assets

- Market value of all assets

Liabilities

- Economic capital
 - i.e. SCR + Free Surplus
- Non-hedgeable liabilities
 - Best Estimate
 - Risk Margin
- Hedgeable liabilities
 - Market-Consistent Value

SCR in a Simplified Internal Model

- **Time:** $t = 0, 1$
- **Value of assets:** $A_t, t = 0, 1$
- **Value of liabilities:** $L_t, t = 0, 1$
- **Capital (NAV):** $E_t = A_t - L_t, t = 0, 1$

$$P(E_1 \leq 0) \leq \alpha$$

$$\Leftrightarrow E_1 \in \mathcal{A}_{V@R_\alpha}$$

$$\Leftrightarrow SCR := V@R_\alpha(\Delta A_1 - \Delta L_1) \leq E_0,$$

with $\Delta A_1 = A_1 - A_0, \Delta L_1 = L_1 - L_0$.

Alternative Risk Measures

- Model for one time period as in Solvency II: $t = 0, 1$
- \mathcal{X} is space of positions at time 1 modeled by random variables (P&L)

Risk measures

$$\rho : \mathcal{X} \rightarrow \mathbb{R}$$

- **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- **Cash invariance:** If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

A risk measure is a statistics that summarizes certain properties of random future balance sheets.

Risk measures like V@R focus on the downside risk.

Capital requirements

- A position $X \in \mathcal{X}$ is **acceptable**, if $\rho(X) \leq 0$.
The collection \mathcal{A} of all acceptable positions is the **acceptance set**.
- ρ is a **capital requirement**, i.e.

$$\rho(X) = \inf \{m \in \mathbb{R} : X + m \in \mathcal{A}\}.$$

Example

$$\text{V@R}_\lambda(X) = \inf \{m \in \mathbb{R} : P[m + X < 0] \leq \lambda\}$$

“Smallest monetary amount that needs to be added to a position such that the probability of a loss becomes smaller than λ .”

Value at Risk in the Media

“[David Einhorn](#), who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was

'relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.' ”

“[Nicholas Taleb](#), the best-selling author of 'The Black Swan,' has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' ”

(“Risk Mismanagement”, New York Times, 2. January 2009)

Diversification

Semiconvexity:

$$\rho(\alpha X + (1 - \alpha)Y) \leq \max(\rho(X), \rho(Y)) \quad (\alpha \in [0, 1]).$$

\implies

Convexity (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1 - \alpha)Y) \leq \alpha \rho(X) + (1 - \alpha) \rho(Y) \quad (\alpha \in [0, 1]).$$

Positive homogeneity:

$$\rho(\lambda X) = \lambda \rho(X) \quad (\lambda \geq 0).$$

Geometric properties of the acceptance set

- ρ convex $\Leftrightarrow \mathcal{A}$ convex.
- ρ positively homogeneous $\Leftrightarrow \mathcal{A}$ cone.

Average Value at Risk

$$AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(X) d\gamma$$

Properties

- coherent (i.e. convex and positively homogeneous)
- sensitive to large losses
- basis of Swiss Solvency test and Basel III
- common alternative to V@R in practice
- distribution-based and continuous from above
- building block of large class of risk measures

Utility-based Shortfall Risk (UBSR)

$\ell : \mathbb{R} \rightarrow \mathbb{R}$ convex loss function, z interior point of the range of ℓ .

The **acceptance set** is defined as

$$\mathcal{A} = \{X \in L^\infty : E_P [\ell(-X)] \leq z\}$$

\mathcal{A} induces a **convex** risk measure ρ :

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$$

Simple formula

Shortfall risk $\rho(X)$ is given by the **unique root** s_* of the function

$$f(s) := E[\ell(-X - s)] - z.$$

Utility-based Shortfall Risk (UBSR)

Properties

- convex
- sensitive to large losses
- distribution-based and continuous from above
- easy to estimate and implement
- elicitable

Special case: Expectiles

- Expectiles are precisely the class of coherent UBSRs (W., 2006).
- Acceptability means that the **ratio** of expected gains and expected losses is larger than some given threshold.

Formal definition of expectiles

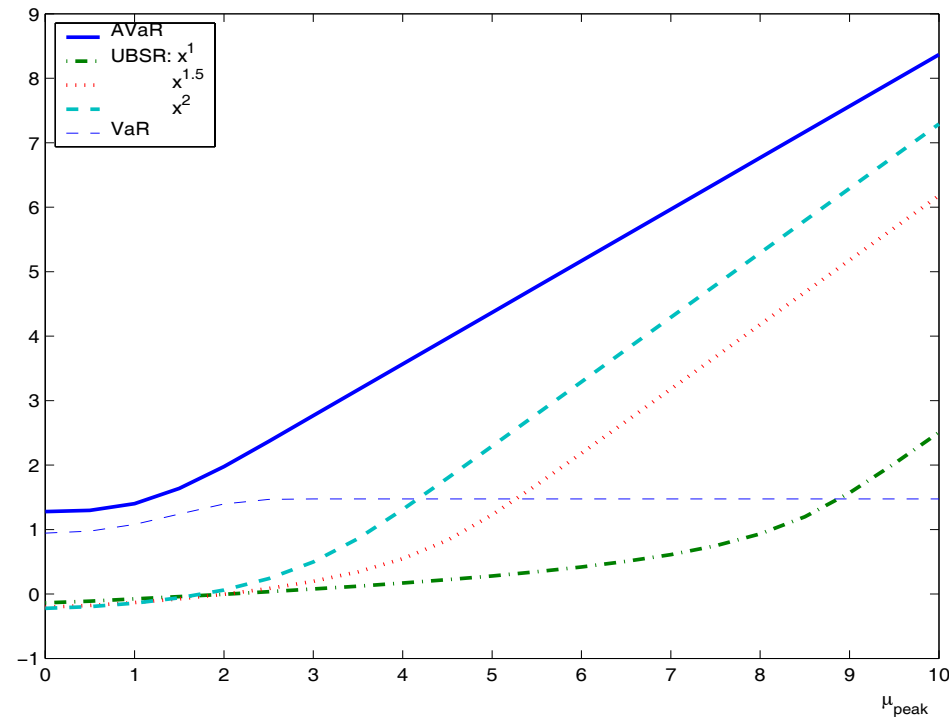
The **acceptance set** is defined as

$$\mathcal{A} = \left\{ X \in L^\infty : \frac{\mathbf{E}_{\mathbf{P}} [X^+]}{\mathbf{E}_{\mathbf{P}} [X^-]} \geq \gamma \right\}$$

\mathcal{A} induces a **coherent** risk measure ρ :

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\}$$

Sensitivity to the Downside Risk



$V@R_{0.05}$, $AV@R_{0.05}$ and UBSR with $p \in \{1, \frac{3}{2}, 2\}$ and $z = 0.3$ of a mixture of a Student t (weight 0.96) and a Gaussian with mean μ (weight 0.04) as a function of μ .

Application to SCR

- **Time:** $t = 0, 1$
- **Value of assets:** $A_t, t = 0, 1$
- **Value of liabilities:** $L_t, t = 0, 1$
- **Capital (NAV):** $E_t = A_t - L_t, t = 0, 1$

$$\begin{aligned} & \rho(E_1) \leq 0 \\ \Leftrightarrow & E_1 \in \mathcal{A}_\rho \\ \Leftrightarrow & \mathbf{SCR} := \rho(\Delta \mathbf{A}_1 - \Delta \mathbf{L}_1) \leq \mathbf{E}_0, \end{aligned}$$

with $\Delta A_1 = A_1 - A_0, \Delta L_1 = L_1 - L_0$.

Examples

Suitable risk measures are e.g. AV@R (SST & Basel III) and expectiles.

Back to Part II: Measures of Systemic Risk

(i) Measures of systemic risk

- General definition on the basis of acceptance sets and system models

(ii) Cash-flow and value models

- Aggregation functions
- Connection to network models

The Basic Ingredients

Consider a one-period economy with n entities.

(i) Cash-flow or value model (CVM)

- $Y = (Y_k)_{k \in \mathbb{R}^n}$ non-decreasing random field
 - For each capital allocation $k = (k_i)_{i=1,2,\dots,n}$ the random variable Y_k captures the relevant stochastic outcome
 - The topological vector space of suitable random variables is denoted by \mathcal{X}

(ii) Objectives of a financial regulator

- $\mathcal{A} \subseteq \mathcal{X}$ set of random variables
 - Each element of \mathcal{A} is acceptable from the point of view of a regulatory authority
 - Mathematically: an acceptance set of a scalar monetary risk measure

Systemic Risk Measures – Definition

Systemic risk is measured by the set of allocations of additional capital that lead to acceptable outcomes.

Definition 1

Letting $\mathcal{P}(\mathbb{R}^n; \mathbb{R}_+^n) := \{B \subseteq \mathbb{R}^n \mid B = B + \mathbb{R}_+^n\}$ be the collection of upper sets with ordering cone \mathbb{R}_+^n , we call the function

$$R : \mathcal{Y} \times \mathbb{R}^n \rightarrow \mathcal{P}(\mathbb{R}^n; \mathbb{R}_+^n)$$

a **systemic risk measure**, if for some acceptance set $\mathcal{A} \subseteq \mathcal{X}$ of a scalar monetary risk measure:

$$\mathbf{R}(\mathbf{Y}; \mathbf{k}) = \{\mathbf{m} \in \mathbb{R}^n \mid \mathbf{Y}_{\mathbf{k}+\mathbf{m}} \in \mathcal{A}\}.$$

Systemic Risk Measures

Elementary Properties

- (i) **Cash-invariance:** $R(Y; k) + m = R(Y; k - m)$
- (ii) **Monotonicity:** $Y \geq Z \Rightarrow (\forall k \in \mathbb{R}^n : R(Y; k) \supseteq R(Z; k))$

→

Further Results

Diversification, robust representations, efficient cash-invariant allocation rules, algorithms,...

Related Literature

- Feinstein, Rudloff & W. (2015)
- Biagini, Fouque, Frittelli & Meyer-Brandis (2015)

Simple Examples of CVMs

Deterministic Transformations

- $X \in L^0(\Omega; \mathbb{R}^n)$ factor vector
- $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}$ increasing aggregation function

\implies

(i) **CVM insensitive to capital levels:** $Y_k = \Lambda(X) + \sum_{i=1}^n k_i$

(ii) **CVM sensitive to capital levels:** $Y_k = \Lambda(X + k)$

Remark: CVMs are not necessarily deterministic transformations of a finite-dimensional factor vector, see Feinstein, Rudloff & W. (2015)

Simple Examples of CVMs (2)

Ad hoc choices of aggregation functions

- (i) System-wide profits and losses: $\Lambda_{\text{sum}}(\mathbf{x}) = \sum_{i=1}^n \mathbf{x}_i$
- (ii) System-wide losses: $\Lambda_{\text{loss}}(\mathbf{x}) = \sum_{i=1}^n -\mathbf{x}_i^-$
- (iii) Multivariate shortfall risk: $\Lambda_{\text{SR}}(\mathbf{x}) = -\ell(-\mathbf{x})$ for some multi-variate loss function ℓ as in Armenti, Crépey, Drapeau & Papapantoleon (2016)

Endogenous aggregation functions

- Typically, the relevant input-output-mechanisms of systems are more complicated than these ad hoc choices.
- This might require a **derivation of aggregation functions** or, more generally, **of CVMs** in financial network models.
- A simple example of a CVM in a network system might take the form $\mathbf{Y}_k = \Lambda(\mathbf{X}, \mathbf{k})$

Simple Examples of CVMs (3)

Consider the comprehensive model of a financial network from the first part of the talk.

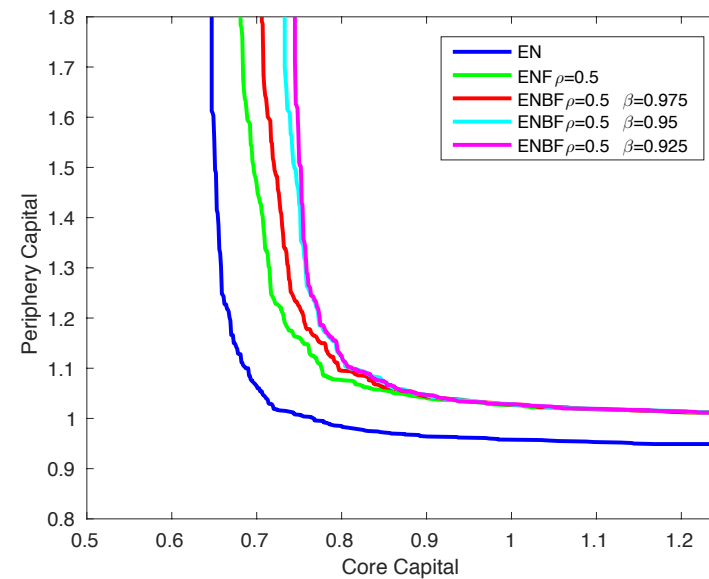
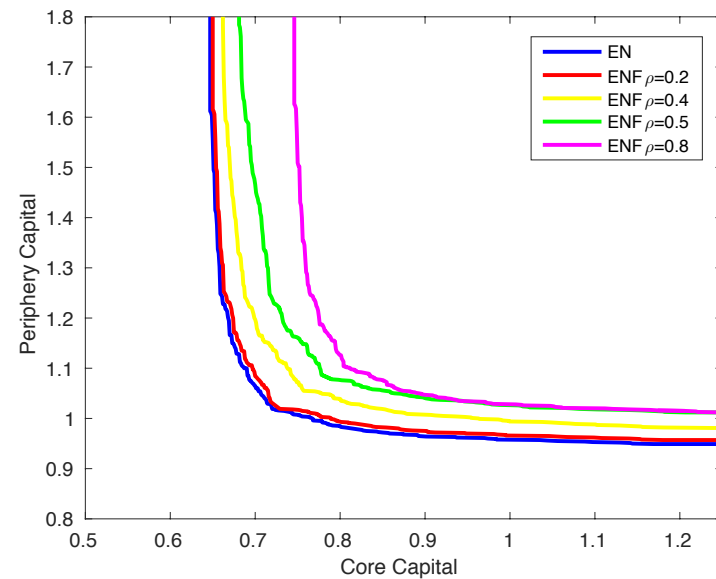
- $k \in \mathbb{R}^n$ additional capital in the banking system
- allocated to the random cash assets $r \in L^0(\Omega; \mathbb{R}^n)$ and the random illiquid assets $s \in L^0(\Omega; \mathbb{R}^n)$ according to some rule
- $(p^*(k), q^*(k))$ is corresponding random largest clearing equilibrium which can be computed scenario-wise

\Rightarrow

The **total losses on external liabilities** are given by

$$Y_k = \sum_{i=1}^n (1 - \sum_{j=1}^n \Pi_{ij}) \cdot (p_i^*(k) - \bar{p}_i)$$

Measures of Systemic Risk



Capital requirements for core and periphery banks as a function of fire sales.

Efficient Allocation Rules

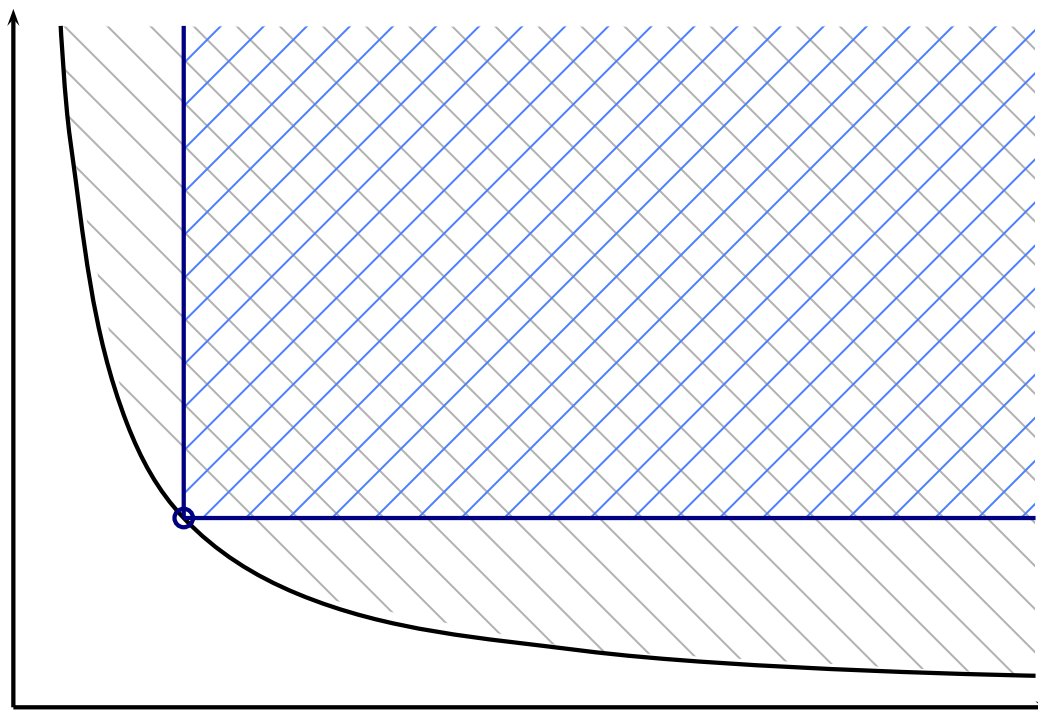


Figure 1: Illustration of a minimal point k^* of an upper set with the orthant $k^* + \mathbb{R}_+^2$ in blue.

Efficient Allocation Rules (2)

Many proposed risk measures from the literature are **special cases** of the cost of efficient allocation rules:

- Chen, Iyengar & Moallemi (2013), see also Kromer, Overbeck & Zilch (2016)
- Brunnermeier & Cheridito (2013)
- Biagini, Fouque, Frittelli & Meyer-Brandis (2015)
- Armenti, Crépey, Drapeau & Papapantoleon (2016)

In addition, CoV@R of Adrian & Brunnermeier (2016) can be reformulated as a special case of our systemic risk measure.

Efficient Allocation Rules – Definition

Definition 2 Let $\mathcal{P}(\mathbb{R}^l)$ be the power set of \mathbb{R}^l . A mapping $k^* : \mathcal{Y} \times \mathbb{R}^l \rightarrow \mathcal{P}(\mathbb{R}^l)$ is called a cash-invariant efficient allocation rules (EAR) associated with a systemic risk measure R , if the following properties are satisfied:

(i) **Minimal values:**

$$k^*(Y; k) \subseteq \text{Min}R(Y; k)$$

(ii) **Convex values:**

$$k^1, k^2 \in k^*(Y; k) \Rightarrow \alpha k^1 + (1 - \alpha)k^2 \in k^*(Y; k)$$

(iii) **Cash-invariance:**

$$k^*(Y; k) + m = k^*(Y; k - m)$$

Efficient Allocation Rules – Characterization

Lemma 1 *Let $R : \mathcal{Y} \times \mathbb{R}^l \rightarrow \mathcal{P}(\mathbb{R}^l; \mathbb{R}_+^l)$ be a systemic risk measure with convex values. For $w : \mathcal{Y} \rightarrow \mathbb{R}_{++}^l$ such that $w(Y) \in \text{recc}R(Y; 0)^+$, the set-valued mapping*

$$\hat{k}(Y; k) = \arg \min \left\{ \sum_{i=1}^l w(Y)_i m_i \mid m \in R(Y; k) \right\} \quad (1)$$

*defines an **EAR**.*

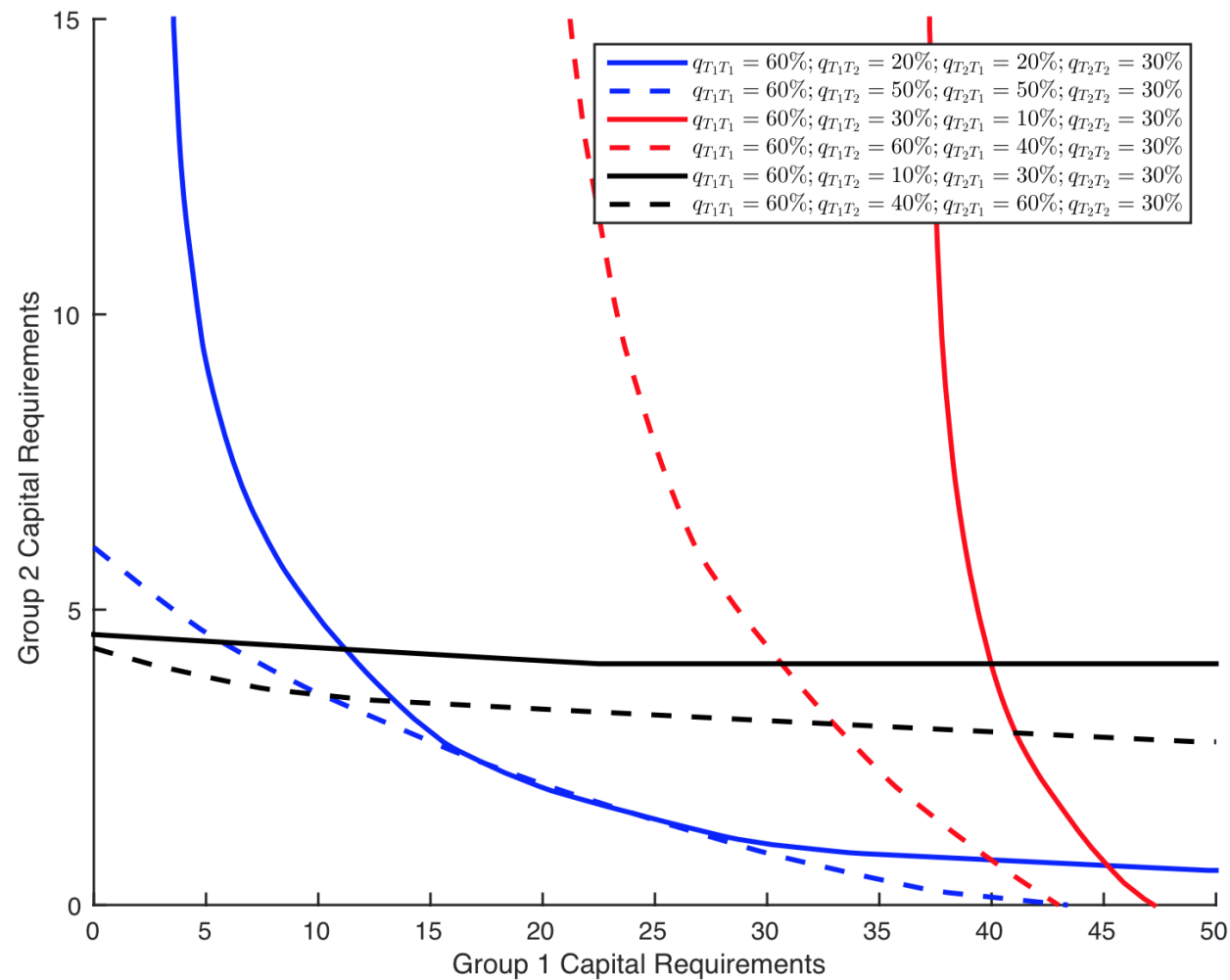
All EARs k^ as defined above are included in EARs \hat{k} of form (1), i.e. $k^*(Y; k) \subseteq \hat{k}(Y; k)$ for all $Y \in \mathcal{Y}$ and $k \in \mathbb{R}^l$.*

- The lemma provides **examples of EARs via a specific choice of the “regulatory price of capital” w .**

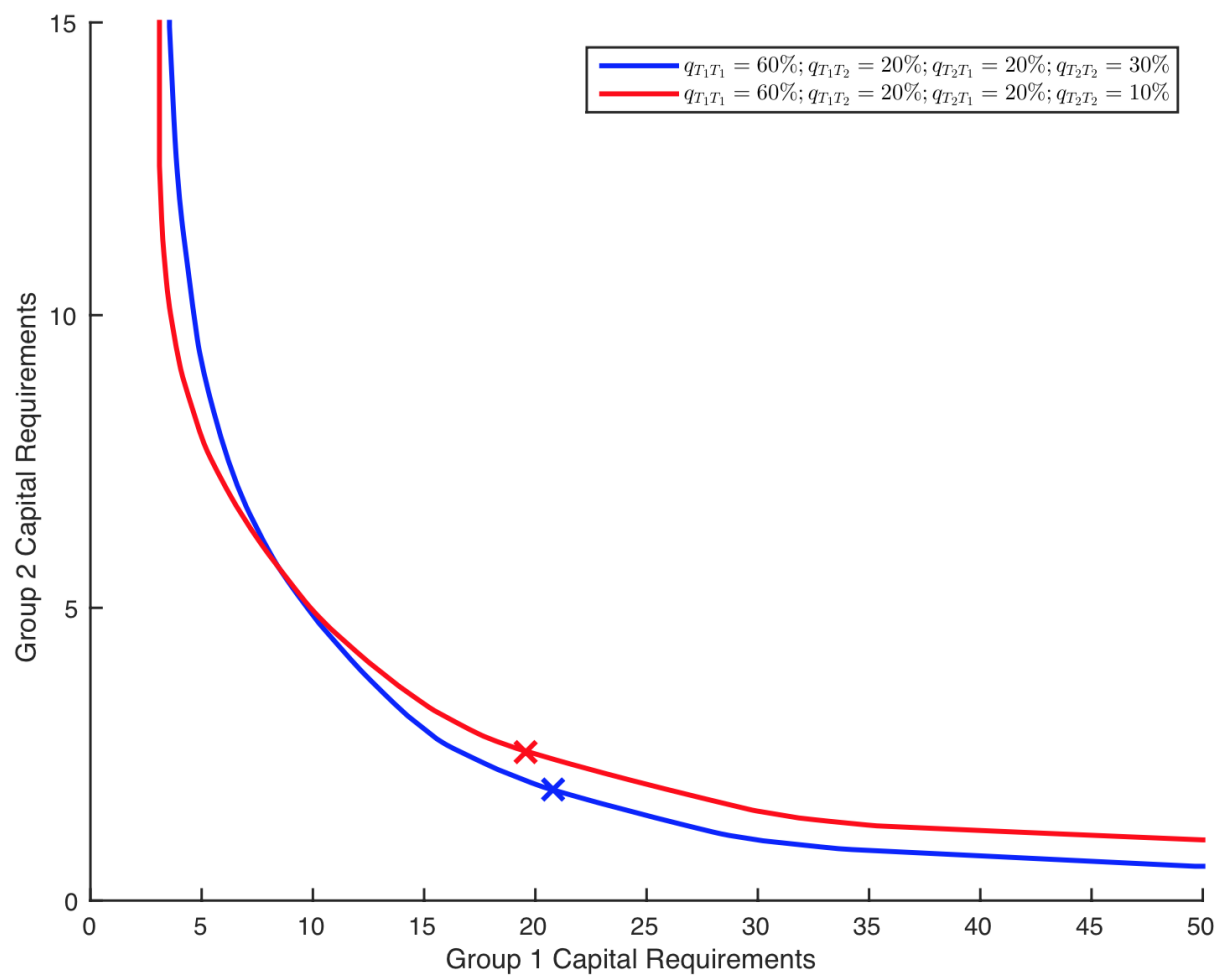
Case Study

- Framework of Eisenberg & Noe (2001):
 - only local interaction in the network
- Tiered graph:
 - Connections are randomly generated, probabilities within tiers and between tiers are fixed
 - Size of obligations within tiers and between tiers along connections are fixed
- 2 Tiers/Groups: few firms with large obligations, many firms with small obligations
- Further ingredients:
 - random endowments, acceptance set defined by $AV@R$
- **Comparative statics: varying the degrees of connectedness**

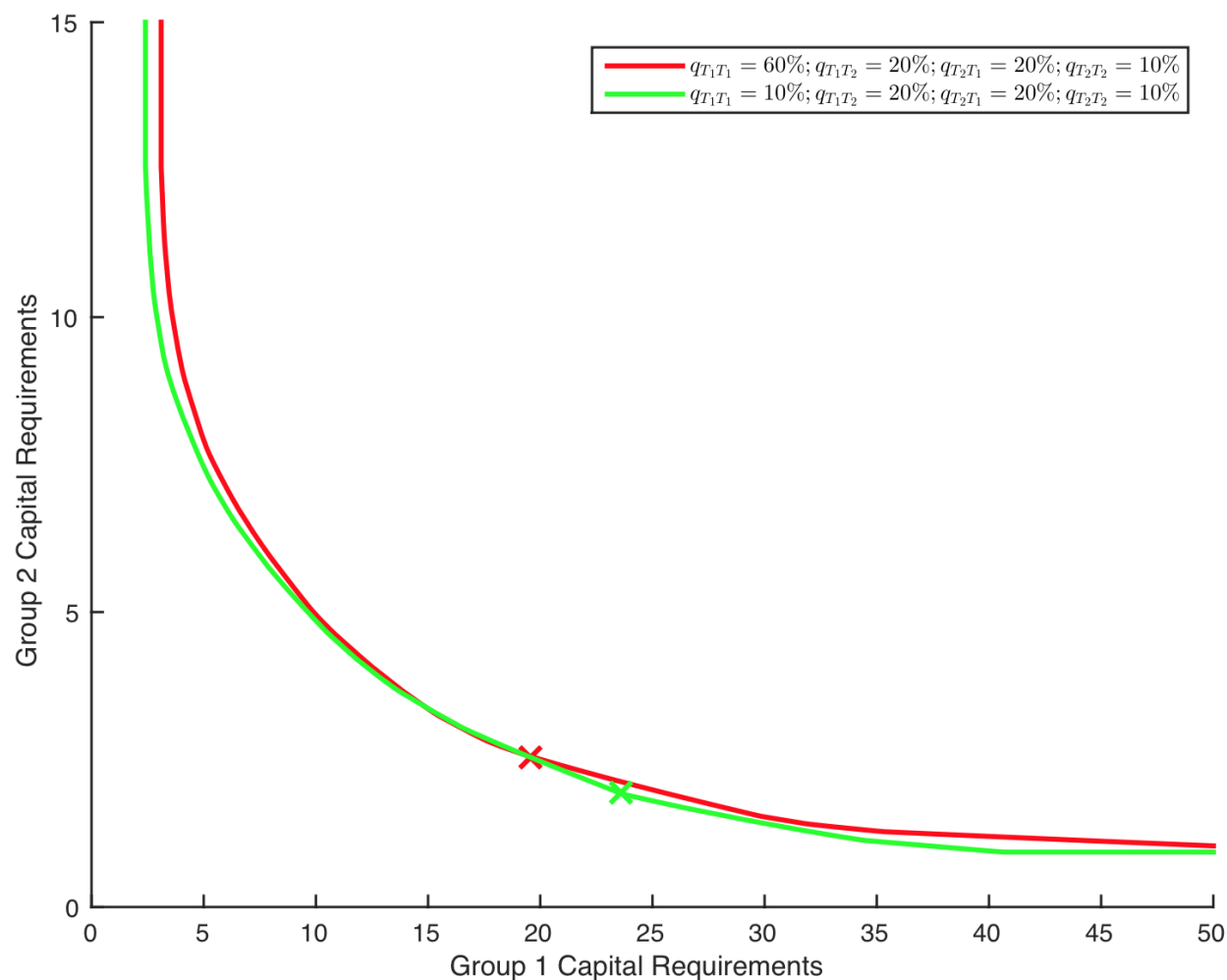
Fixed Intra-Group Connections



Fixed Inter-Group Connections



Fixed Inter-Group Connections



Summary

(i) Comprehensive network model

- Integrates direct liabilities, fire sales, cross holdings and bankruptcy costs
- Toy model for testing regulatory policies and their robustness

(ii) Multi-variate approach to systemic risk

- Proper risk measure instead of default counts
- Based on CVM and acceptances set
- General framework that covers many contributions from the literature
- Ideas can be applied to areas outside finance

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